Geometrical properties of Riemannian superspaces, observables and physical states.*

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Abstract

Classical and quantum aspects of physical systems that can be described by Riemannian non degenerate superspaces are analyzed from the topological and geometrical points of view. For the $N = 1$ case the simplest supermetric introduced in [Physics Letters B **661**, (2008),186] have the correct number of degrees of freedom for the fermion fields and the super-momentum fulfil the mass shell condition, in sharp contrast with other cases in the literature where the supermetric is degenerate. This fact leads a deviation of the 4-impulse (e.g. mass constraint) that can be mechanically interpreted as a modification of the Newton's law. Quantum aspects of the physical states and the basic states and the projection relation between them, are completely described due the introduction of a new Majorana-Weyl representation of the generators of the underlying group manifold. A new oscillatory fermionic effect in the B_0 part of the vaccum solution involving the chiral and antichiral components of this Majorana bispinor is explicitly shown.

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I. INTRODUCTION

The problem of giving an unambiguous quantum mechanical description of a particle in a general spacetime has been repeatedly investigated. The introduction of supersymmetry provided a new approach to this question, however, some important aspects concerning the physical observables remain not completely understood, classically and quantically speaking.

The superspace concept, on the other hand, simplify considerably the link between ordinary relativistic systems and "supersystems", extending an standard (bosonic) spacetime by means of a general (super)group manifold, equipped with also fermionic (odd) coordinates.

In a previous reference [2] we introduced, besides other supersymmetric quantum systems of physical interest, a particular $N = 1$ superspace [3], with the aim of studying a superworld-line quantum particle (analogously to the relativistic one) and its relation with SUGRA theories [1,2,10]. The main feature of this superspace is that the supermetric is *invertible and non-degenerate*, and is the basic ingredient of a Volkov-Pashnev particle action [3] that is, of type $G4$ in the Casalbuoni's classification [4]. As shown in [2,10], the non-degeneracy of the supermetrics (and therefore the corresponding superspaces) in the description of physical systems leads to important geometrical and topological effects on the quantum states, namely, *consistent mechanisms of localization and confinement*, due purely to the geometrical character of the Lagrangian. Also an *alternative to the Randall-Sundrum(RS)* model without extra bosonic coordinates can be consistently formulated, eliminating the problems that the RS-like models present at the quantum level[2].

Given the importance of the non degeneracy of the supermetrics in the formulation of physical theories, in the present letter we analyze further the super-line element introduced in [2,3] in several senses. In Section II, classical aspects of physical observables arising from the geometrical properties of the superspace are derived, analyzed and compared with the results of well known models where the metric is degenerate. Section III is devoted to describe the bosonic B_0 part of the solution states and the corresponding probability currents (j_0), in the temporal and geometrical degeneracy limits ($t, a \to 0, \infty$). In Section IV, a *new Majorana-Weyl representation* for the generators of the metaplectic group[9] is introduced. The reduction from the 6 dimensional group manifold to the 4 dimensional spacetime is explicitly given and the specific action of the generators of the metaplectic group on the physical and basic states, is performed. Finally, in Section V, the concluding remarks and comments on the main results are given.

II. FERMIONIC MOMENTUM AND SUPERVELOCITY: PHYSICAL CONSTRAINTS ON THE SUPERPARTICLE ACTION

To begin with, we consider the line element of the non-degenerated supermetric [2,3] introduced in [1]

$$ds^2 = \omega^\mu \omega_\mu + \mathbf{a}\omega^\alpha \omega_\alpha - \mathbf{a}^* \omega^{\dot{\alpha}} \omega_{\dot{\alpha}}, \tag{1}$$

where the bosonic term and the Majorana bispinor compose a superspace $(1,3|1)$, with coordinates $(t, x^i, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, and where Cartan forms of supersymmetry group are described by

$$\omega_\mu = dx_\mu - i(d\theta\sigma_\mu\bar{\theta} - \theta\sigma_\mu d\bar{\theta}), \qquad \omega^\alpha = d\theta^\alpha, \qquad \omega^{\dot{\alpha}} = d\theta^{\dot{\alpha}} \tag{2}$$

(obeying evident supertranslational invariance). As we have extended our manifold to include fermionic coordinates, it is natural to extend also the concept of trajectory of point particle to the superspace. To do this we take the coordinates $x(\tau)$, $\theta^\alpha(\tau)$ and $\overline{\theta}^{\dot{\alpha}}(\tau)$ depending on the evolution parameter τ. Geometrically, the action functional that will describe the world-line of the superparticle reads

$$S = \int_{\tau_1}^{\tau_2} d\tau L\left(x,\theta,\overline{\theta}\right) = -m\int_{\tau_1}^{\tau_2} d\tau \sqrt{\overset{\circ}{\omega}_\mu \overset{\circ}{\omega}{}^\mu + \mathbf{a}\dot{\theta}^\alpha \dot{\theta}_\alpha - \mathbf{a}^* \dot{\overline{\theta}}{}^{\dot{\alpha}} \dot{\overline{\theta}}_{\dot{\alpha}}} \tag{3}$$

where $\overset{\circ}{\omega}_\mu = \dot{x}_\mu - i(\dot{\theta}\ \sigma_\mu\overline{\theta} - \theta\ \sigma_\mu\dot{\overline{\theta}})$, and the dot indicates derivative with respect to the parameter τ, as usual.

The momenta, canonically conjugated to the coordinates of the superparticle, are

$$\begin{aligned}
\mathcal{P}_\mu &= \partial L/\partial x^\mu = (m^2/L)\,\overset{\circ}{\omega}_\mu \\
\mathcal{P}_\alpha &= \partial L/\partial\dot{\theta}^\alpha = i\mathcal{P}_\mu\left(\sigma^\mu\right)_{\alpha\dot{\beta}}\dot{\overline{\theta}}{}^{\dot{\beta}} + (m^2\mathbf{a}/L)\,\dot{\theta}_\alpha \\
\mathcal{P}_{\dot{\alpha}} &= \partial L/\partial\dot{\overline{\theta}}{}^{\dot{\alpha}} = i\mathcal{P}_\mu\theta^\alpha\left(\sigma^\mu\right)_{\alpha\dot{\alpha}} - (m^2\mathbf{a}^*/L)\,\dot{\overline{\theta}}_{\dot{\alpha}}
\end{aligned} \tag{4}$$

Notice the first important fact: the fermionic momenta $\mathcal{P}_\alpha$ and $\mathcal{P}_{\dot{\alpha}}$ are proportional to $\dot{\theta}_\alpha$ and $\dot{\overline{\theta}}_{\dot{\alpha}}$ due to $\mathbf{a}$ and $\mathbf{a}^*$. That is not the case in the standard superparticle actions where the superspace metric is degenerate, *e.g.* without these complex coefficients. This is the case, for instance, of the constrained Brink-Schwartz superparticle model or the relativistic $G4$ of Casalbuoni (see [4,5,6]), where the Lagrangian presents only terms of the form $\sim \overline{\theta}^{\dot{\alpha}}\dot{\overline{\theta}}_{\dot{\alpha}}, \theta^\alpha\dot{\theta}_\alpha$ etc., which leads to fermionic momenta proportional to the coordinates *e.g.* $\mathcal{P}_{\dot{\alpha}} = \partial L/\partial\dot{\overline{\theta}}{}^{\dot{\alpha}} \propto \overline{\theta}_{\dot{\alpha}}$. This reduces, consequently, the number of fermionic degrees of freedom to n instead of the $2n$ logically required.

It is difficult to study this system in a Hamiltonian formalism framework because of the constraints and the nullification of the Hamiltonian. As the action (3) is invariant under reparametrizations of the evolution parameter $(\tau \to \widetilde{\tau} = f(\tau))$, one way to overcome this difficulty is to identify the dynamical variable x_0 with the time. It is sufficient to introduce the concepts of integration and derivation in supermanifolds, as we have done in [1], to have

the action rewritten in the form

$$S=-m\int_{\tau 1}^{\tau 2}\dot{x}_0 d\tau\sqrt{\left[1-iW_{,0}^{0}\right]^2-\left[x_{,0}^{i}-W_{,0}^{i}\right]^2+\dot{x}_0^{-2}\left(\mathbf{a}\dot{\theta}_\alpha\dot{\theta}^\alpha-\mathbf{a}^*\dot{\overline{\theta}}_{\dot{\alpha}}\dot{\overline{\theta}}^{\dot{\alpha}}\right)} \tag{5}$$

where the $W_{,0}^{\mu}$ are defined by

$$\overset{\circ}{\omega}{}^{0}=\dot{x}^{0}\left[1-iW_{,0}^{0}\right],\qquad \overset{\circ}{\omega}{}^{i}=\dot{x}^{0}\left[x_{,0}^{i}-iW_{,0}^{i}\right] \tag{6}$$

If $x_0(\tau)$ is taken to be the evolution parameter, then

$$S=-m\int_{x_0(\tau_1)}^{x_0(\tau_2)}dx_0\sqrt{\left[1-iW_{,0}^{0}\right]^2-\left[x_{,0}^{i}-W_{,0}^{i}\right]^2+\mathbf{a}\dot{\theta}^{\alpha}\dot{\theta}_\alpha-\mathbf{a}^*\dot{\overline{\theta}}^{\dot{\alpha}}\dot{\overline{\theta}}_{\dot{\alpha}}}\equiv\int dx_0 L \tag{7}$$

Physically, this 'dynamical parameter' x_0 corresponds to the time measured by an observer's clock in the rest frame.

The total relativistic velocity in the superspace (supervelocity) can be derived as usual from the line element of the supermetric, using a parameter of evolution of the physical system τ. Then we have, from (1), the 'true' supervelocity

$$\begin{aligned}\dot{z}^{A}\dot{z}_A\equiv v^2=\left(\frac{ds}{d\tau}\right)^2 &= \dot{x}^{\mu}\dot{x}_{\mu}-2i\dot{x}^{\mu}(\dot{\theta}\sigma_\mu\bar{\theta}-\theta\sigma_\mu\dot{\bar{\theta}}) \\ &+\left(\mathbf{a}-\bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}}\right)\dot{\theta}^{\alpha}\dot{\theta}_\alpha-\left(\mathbf{a}^*+\theta^\alpha\theta_\alpha\right)\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}\end{aligned} \tag{8}$$

with $z_A\equiv\left(x_\mu,\theta_\alpha,\overline{\theta}_{\dot{\alpha}}\right)$.

Note that in the case of [4], due to the degeneracy of the superspace metric, there is not supervelocity but the usual 4-velocity, as in the pure relativistic case. There is also another notorious difference between the 4-velocity and the supervelocity, precisely coming from the classical relativistic theory: in the relativistic case the 4-velocity fulfils ($\hbar=c=1$)

$$v_{rel}^2=\dot{x}^{\mu}\dot{x}_{\mu}=-1 \tag{9}$$

in conicidence with the mass shell condition from the impulses $p_\mu p^\mu=m^2$, as in the original $G4$ model [4].

In our case, due to the non degenerate super-line element (1), we have the mass shell condition $p_A p^A=m^2$, with the A index taking values on $(x,\theta,\overline{\theta})$. Then, the 'supervelocity' expression (8) automatically fulfils

$$\begin{aligned}\dot{z}^{A}\dot{z}_A=-1 &= \dot{x}^{\mu}\dot{x}_{\mu}-2i\dot{x}^{\mu}(\dot{\theta}\sigma_\mu\bar{\theta}-\theta\sigma_\mu\dot{\bar{\theta}}) \\ &+\left(\mathbf{a}-\bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}}\right)\dot{\theta}^{\alpha}\dot{\theta}_\alpha-\left(\mathbf{a}^*+\theta^\alpha\theta_\alpha\right)\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}\end{aligned} \tag{10}$$

i.e. our supersymmetric model is enforced to accomplish the standard classical relativistic conditions.

A. Some particular cases

i) It is not difficult to see that, in the case when the Majorana spinors are null, or constant with respect to the evolution parameter of the system (or proper time), we have ($c = 1$)

$$v^2\big|_{\theta,\bar{\theta}=cte} = \dot{x}^\mu \dot{x}_\mu = -1. \tag{11}$$

That is, the system velocity is given by a pure bosonic contribution and we recover the standard expression, due to (10). Notice that the relativistic condition on the super-velocity will bring us several constraints between bosonic and fermionic coordinates of the supersymmetric system.

ii) Now, if we consider the bosonic variables constant with respect to the evolution parameter of the system, we arrive at

$$v^2\big|_{x=cte} = -1 = \left(\mathbf{a}-\bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}}\right)\dot{\theta}^\alpha\dot{\theta}_\alpha - \left(\mathbf{a}^* + \theta^\alpha\theta_\alpha\right)\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}, \tag{12}$$

such that the kinetic term is due to Majorana bispinors. However, if the time parameter is identified with $d\tau = dt$ (proper time), the relativistic velocity for the static case (free fall) lead us to the following constraint

$$\begin{aligned} v^2\big|_{\tau=x_0} &\rightarrow 2i(\dot{\theta}\sigma_0\bar{\theta} - \theta\sigma_0\dot{\bar{\theta}}) \\ &= \left(\mathbf{a}-\bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}}\right)\dot{\theta}^\alpha\dot{\theta}_\alpha - \left(\mathbf{a}^* + \theta^\alpha\theta_\alpha\right)\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}. \end{aligned} \tag{13}$$

In the case of Casalbuoni's superparticle action ($\mathbf{a} = \mathbf{a}^* = 0$), the expression (10) is reduced to

$$v^2\big|_{\mathbf{a}=\mathbf{a}^*=0} = \left(\frac{ds}{d\tau}\right)^2 = \dot{x}^\mu\dot{x}_\mu - 2i\dot{x}^\mu(\dot{\theta}\sigma_\mu\bar{\theta} - \theta\sigma_\mu\dot{\bar{\theta}}) + \left(\bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}}\right)\left(\dot{\theta}^\alpha\dot{\theta}_\alpha\right) - \left(\theta^\alpha\theta_\alpha\right)\left(\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}\right) \tag{14}$$

Passing to the proper time and free fall cases as before, we arrive to the important conclusion that in the Casalbuoni's model

$$v^2\big|_{\tau=x_0} \rightarrow 2i(\dot{\theta}\sigma_0\bar{\theta} - \theta\sigma_0\dot{\bar{\theta}}) \equiv \left(\bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}}\right)\left(\dot{\theta}^\alpha\dot{\theta}_\alpha\right) - \left(\theta^\alpha\theta_\alpha\right)\left(\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}\right) \tag{15}$$

In order to fulfil this condition, the fermions shall be null or constant with respect to the evolution parameter of the system. This is a direct consequence of the degeneration (non invertibility) of the supermetric in the Brink-Schwartz or the Casalbuoni-$G4$ type cases.

This inconsistence of supersymmetric models based on degenerate supermetrics is translated into the incompatibility of the 'natural constraints' for the relativistic conditions on the corresponding superline elements.

In the full super-relativistic case, however, we obtain the mass shell condition $p_A p^A = m^2$, with $A = (x, \theta, \overline{\theta})$. But, in sharp contrast with the super-line element used by Casalbuoni, the square of the 4-impulse is not m^2, as is easily seen from the expressions (4). This issue will be treated deeply in [7].

In summary: non degenerate (invertible) supermetrics basis of relativistic $G4$ models lead to consistent SUSY analogs of (pseudo)classical relativistic systems, in the sense that the phase space number of degrees of freedom is the correct ones. Therefore, the exact quantization of the system can be performed by standard procedures: canonical (Gupta-Bleuler in the case of QFT, etc.)[3,6] or non-canonical ones (group theoretical, geometrical, coherent states quantizations, etc.)[1,2,9].

Also the 4-momentum in the case of the non degenerate superspace does not fulfil the mass shell condition. This fact can be (classically) translated to an observable deviation from the Newton's law of gravitation. In the next sections the meaning of these constraints and the possible explanation for deviations of the GR predictions will be briefly discussed.

III. ON THE B_0 PART OF THE SPINORIAL SOLUTION

From previous works [1, 2], the supermultiplet solution for the geometric lagrangian is

$$\begin{aligned} g_{ab}(0,\lambda) &= \langle\psi_\lambda(t)|\, L_{ab}\, |\psi_\lambda(t)\rangle \\ &= e^{-\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2} e^{\xi\varrho(t)} \chi_f \langle\psi_\lambda(0)| \begin{pmatrix} a \\ a^\dagger \end{pmatrix}_{ab} |\psi_\lambda(0)\rangle \end{aligned} \tag{16}$$

Consider, for simplicity, the 'square' solution for the three compactified dimensions [2] (spin λ fixed)

$$g_{ab}(t) = e^{A(t)+\xi(\phi_\alpha(t)+\bar{\chi}_{\dot{\alpha}}(t))} g_{ab}(0) \tag{17}$$

We have obtained schematically for the exponential fermionic part

$$\begin{aligned} \varrho(t) \equiv \overset{\circ}{\phi}_\alpha \Big[\big(\alpha e^{i\omega t/2} &+ \beta e^{-i\omega t/2}\big) - \left(\sigma^0\right)^{\alpha}_{\dot{\alpha}} \big(\alpha e^{i\omega t/2} - \beta e^{-i\omega t/2}\big) \Big] \\ &+ \frac{2i}{\omega} \left[\left(\sigma^0\right)_{\alpha}^{\ \dot{\beta}} \ \overline{Z}_{\dot{\beta}} + \left(\sigma^0\right)^{\alpha}_{\ \dot{\alpha}} \ Z_\alpha \right] \end{aligned} \tag{18}$$

For the exponential bosonic part we have

$$A(t) = -\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + c_1 t + c_2 \tag{19}$$

And the initial value for the metric is given by

$$g_{ab}(0) = \langle\psi(0)| \begin{pmatrix} a \\ a^\dagger \end{pmatrix}_{ab} |\psi(0)\rangle, \tag{20}$$

where $\overset{\circ}{\phi}_\alpha, Z_\alpha, \overline{Z}_{\dot{\beta}}$ are constant spinors, and α and β are $\mathbb{C}$-numbers. The constant $c_1 \in \mathbb{C}$ due the obvious physical reasons and the chirality restoration of the superfield solution [1,2,10].

Notice that there exists a factor in the vacuum solution eq.(16), that we call χ_f, coming from the odd generators of the big covering group related to the symmetries of the specific model (3). These 'parafermionic' part and the associated odd generators [1,9,10] will not be treated in this paper, and will be left aside.

Two geometric-physical options will be related to the orientability of the superspace trajectory[11]: $\alpha = \pm\beta$. We take, without lose generality $\alpha = +\beta$ then, exactly,

$$\varrho(t) = \begin{pmatrix} \overset{\circ}{\phi}_\alpha \cos(\omega t/2) + \frac{2}{\omega} Z_\alpha \\ -\overset{\circ}{\overline{\phi}}_{\dot{\alpha}} \sin(\omega t/2) - \frac{2}{\omega} \overline{Z}_{\dot{\alpha}} \end{pmatrix} \tag{21}$$

which obviously represents a *Majorana fermion* where the $\mathbb{C}$ symmetry is inside of the constant spinors.

The spinorial part of the superfield solution in the exponent becomes

$$\xi\varrho(t) = \theta^\alpha \left(\overset{\circ}{\phi}_\alpha \cos(\omega t/2) + \frac{2}{\omega} Z_\alpha\right) - \overline{\theta}^{\dot{\alpha}} \left(-\overset{\circ}{\overline{\phi}}_{\dot{\alpha}} \sin(\omega t/2) - \frac{2}{\omega} \overline{Z}_{\dot{\alpha}}\right) \tag{22}$$

In the above expression there appear a type of Zitterbewegung or continous oscillation between the chiral and antichiral part of the bispinor $\varrho(t)$. The physical meaning of such an oscillation is not completely clear for us given that is an effect never described before from the theoretical point of view. Figures 1, 2 and 3 are describing qualitatively such effect for suitable values of the parameters of the vacuum solution and with an increasing ω respectively ($\omega_1 < \omega_2 < \omega_3$) This important issue will be aborded in detail in a separate publication [8].

The interesting point of the physical states (and the basic ones) in explicit form is that their behaviour (and the behavior of the zero component of the current of probability $j_0(t)$),

can be analyzed in the limits of interest. This fact is very important in order to understand deeply the even part B_0 of the superfield solution, then, the fermionic and bosonic evolution of the system.

A. Temporal limits

i) For the square states (observables)

$$g_{ab}(t) = \overbrace{e^{-\left(\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2\right)} e^{\xi\rho(t)}}^{F(t)} \overbrace{\left[\alpha\begin{pmatrix}1\\0\end{pmatrix} + \alpha^*\begin{pmatrix}0\\1\end{pmatrix}\right]}^{v_{ab}} \tag{23}$$

the standard temporal limits at $t=0$ and $t\to\infty$ are

$$g_{ab}(t=0) = F(0)v_{ab}, \qquad F(0) = e^{-c_2} e^{\theta^\alpha \overset{\circ}{\phi}_\alpha + \frac{2}{\omega}\left(\theta^\alpha Z_\alpha + \overline{\theta}^{\dot{\alpha}} \overline{Z}_{\dot{\alpha}}\right)} \tag{24}$$

and

$$g_{ab}(t\to\infty) \to 0 \tag{25}$$

The zero component of the current of probability for this observable reads

$$j_0(t) = 2E^2 g^\dagger_{ab}(t) g^{ab}(t) \tag{26}$$

and therefore is positive definite (the energy appears squared), as was pointed out in [1, 2]. The corresponding temporal limits at $t=0$ and $t\to\infty$ are

$$j(t=0) = 2E^2 g^\dagger_{ab}(0) g^{ab}(0) = 4E^2|\alpha|^2 F^2(0) \tag{27}$$

and

$$j_0(t\to\infty) \to 0 \tag{28}$$

ii) Analogously, for the 'square root' basic (non-observable) states we have

$$\psi(t) = \overbrace{e^{-\frac{1}{2}\left(\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2\right)} e^{\frac{1}{2}\xi\rho(t)}}^{\equiv F^{1/2}(t)} \overbrace{\left[\sqrt{\alpha}\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\alpha^*}\begin{pmatrix}0\\1\end{pmatrix}\right]}^{\equiv v^{1/2}_{ab}} \tag{29}$$

The standard temporal limit at $t=0$ takes the form

$$\psi(0) = F^{1/2}(0) v^{1/2}_{ab} \qquad \text{with} \quad F^{1/2}(0) = e^{-\frac{c_2}{2}} e^{\frac{\theta^\alpha \overset{\circ}{\phi}_\alpha}{2} + \frac{1}{\omega}\left(\theta^\alpha Z_\alpha + \overline{\theta}^{\dot{\alpha}} \overline{Z}_{\dot{\alpha}}\right)} \tag{30}$$

And, at $t \to \infty$, is

$$\psi(t \to \infty) \to 0 \tag{31}$$

The current of probability for the 'square root' states (non-observable)

$$j_0(t) = 2E\psi^\dagger(t)\psi(t) \tag{32}$$

Then, as was pointed out in [1, 2], it is *not* positive definite and the limits at $t = 0$

$$j_0(0) = 2E\psi^\dagger(0)\psi(0) = 4E|\alpha|F^{1/2}(0) \tag{33}$$

and at $t \to \infty$

$$j_0(t \to \infty) \to 0 \tag{34}$$

B. Special limits (m fixed)

1. *Physical states*

Two limiting cases are to be considered

i) Bosonic ultralocalization: $\lim a \to 0$, $\omega \to \infty$, $\xi\varrho(t) \to 0$

$$g_{ab}(t) = \overbrace{e^{-\left(\frac{m}{|a|}\right)^2 t^2}}^{F(t)} v_{ab}, \qquad j_0(t) = 4E^2|\alpha|^2 e^{-\frac{2m}{|a|}^2 t^2} \tag{35}$$

only boson part remains.

ii) Fermionic bosonization: $\lim a \to \infty$, $\omega \to 0$, $\xi\varrho\,(t) \to \theta^\alpha \mathring{\phi}_\alpha + \frac{2}{\omega}\left(\theta^\alpha Z_\alpha + \overline{\theta}^{\dot{\alpha}}\overline{Z}_{\dot{\alpha}}\right)$

$$g_{ab}(t) = \overbrace{e^{-(c_1 t + c_2)} e^{\theta^\alpha \mathring{\phi}_\alpha + \frac{2}{\omega}\left(\theta^\alpha Z_\alpha + \overline{\theta}^{\dot{\alpha}}\overline{Z}_{\dot{\alpha}}\right)}}^{F(t)} v_{ab}, \qquad j_0(t) = 4E^2|\alpha|^2 e^{-2c_2} e^{2\theta^\alpha \mathring{\phi}_\alpha + \frac{4}{\omega}\left(\theta^\alpha Z_\alpha + \overline{\theta}^{\dot{\alpha}}\overline{Z}_{\dot{\alpha}}\right)} \tag{36}$$

only bosonized fermionic part remains at this limit.

2. *Basic states*

i) Bosonic ultralocalization: $\lim a \to 0$, $\omega \to \infty$, $\xi\varrho\,(t) \to 0$

$$\psi(t) = e^{-\frac{1}{2}\left(\frac{m}{|a|}\right)^2 t^2} v_{ab}^{1/2}, \qquad j_0(t) = 4E^2|\alpha|^2 e^{-\frac{m}{|a|}^2 t^2} v_{ab}^{1/2} \tag{37}$$

ii) Fermionic bosonization: Lim $a \to \infty$, $\omega \to 0$, $\xi \varrho(t) \to 0$

$$\psi(t) = e^{-\frac{(c_1 t + c_2)}{2}} e^{\frac{\theta^\alpha \overset{\circ}{\phi}_\alpha}{2} + \frac{1}{\omega}\left(\theta^\alpha Z_\alpha + \overline{\theta}^{\dot{\alpha}} \overline{Z}_{\dot{\alpha}}\right)} v_{ab}^{1/2}, \tag{38}$$

$$j_0(t) = 4E^2 |\alpha|^2 e^{-c_2} e^{\theta^\alpha \overset{\circ}{\phi}_\alpha + \frac{2}{\omega}\left(\theta^\alpha Z_\alpha + \overline{\theta}^{\dot{\alpha}} \overline{Z}_{\dot{\alpha}}\right)} \tag{39}$$

With similar comments than for the physical states. Also, the dynamics of the system trivialize in the limit when $\frac{t}{|\mathbf{a}|} \tilde{} \omega t \to$constant.

It is very important to note that we study specifically the corresponding currents (for the square states and for the basic ones) because the number operator will be related to the Hamiltonian operator for the basic states, and for the square ones that in essence can be different [1,9,10] and main task of a future work [7].

IV. EXOTIC MAJORANA REPRESENTATIONS AND SPINOR TRANSFORMATIONS

Because the underlying non compact symmetry of the Lagrangian corresponds to $Mp(4)$ (metaplectic group in four dimensions), we need to introduce a suitable (matrix) representation of its irreducible part, namely $Mp(2)$: the 3 dimensional metaplectic group. That is such because $Mp(4)$ splits into 2 irreducible $Mp(2)$ subgroups. The $Mp(2)$ is the four covering of the L_3 (the 3-dimensional Lorentz group or *little group*) or the double covering of $SL(2R)$.

The new Majorana-Weyl representation that we introduce below is given by the 2 by 2 following operators

$$\sigma_\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{40}$$

where the required condition over such matrices $\sigma_\alpha \wedge \sigma_\beta = \sigma_\gamma$, $\sigma_\beta \wedge \sigma_\gamma = \sigma_\alpha$ and $\sigma_\gamma \wedge \sigma_\alpha = -\sigma_\beta$, evidently holds (Lie group) given the underlying non-compact symmetry.

Notice, however, that the Majorana-Weyl representation (40) gives, in some sense, the undotted description of the $Mp(4)$. The adjoint part (dotted) is absolutely analog. Then, it is sufficient to analyze the undotted (dotted) part.

Using the new representation given by the matrices (40), the underlying symmetry of the bosonic (even) part of the supermultiplet system, can be manifestly exposed as follows

[1,10]: from the geometrical Lagrangian, we can calculate the non-observable spinor field (analogouslly the respective square g_{ab})

$$\begin{aligned}\psi(t) &= \overbrace{e^{-\frac{1}{2}\left(\left(\frac{m}{|a|}\right)^{2}t^{2}+c_{1}t+c_{2}\right)}e^{\frac{1}{2}\xi\rho(t)}}^{\equiv F^{1/2}(t)}\overbrace{\left[\sqrt{\alpha}\begin{pmatrix}1\\0\end{pmatrix}+\sqrt{\alpha^{*}}\begin{pmatrix}0\\1\end{pmatrix}\right]}^{\equiv v_{ab}^{1/2}},\\ g_{ab}(t) &= \overbrace{e^{-\left(\left(\frac{m}{|a|}\right)^{2}t^{2}+c_{1}t+c_{2}\right)}e^{\xi\rho(t)}}^{F(t)}\overbrace{\left[\alpha\begin{pmatrix}1\\0\end{pmatrix}+\alpha^{*}\begin{pmatrix}0\\1\end{pmatrix}\right]}^{v_{ab}}\end{aligned} \tag{41}$$

where dependence on time is in $F(t)$ (or $F^{1/2}(t)$), and the $\mathbb{C}$ basic spinor was defined as v_{ab} (or $v_{ab}^{1/2}$) (whose meaning in terms of the coherent states will be shown soon).

Also, it is necessary to introduce here the second basic spinor

$$w_{ab}^{1/2}=\left[\sqrt{\alpha}\begin{pmatrix}1\\0\end{pmatrix}-\sqrt{\alpha^{*}}\begin{pmatrix}0\\1\end{pmatrix}\right],\; w_{ab}=\left[\alpha\begin{pmatrix}1\\0\end{pmatrix}-\alpha^{*}\begin{pmatrix}0\\1\end{pmatrix}\right] \tag{42}$$

which is needed for the full description below. Notice that, at this stage, the only necessary spinors in order to describe the dynamical relation between the group manifold, the space-time and the related symmetries involved, are $v_{ab}\left(v_{ab}^{1/2}\right)$, $w_{ab}\left(w_{ab}^{1/2}\right)$ and the generators (matrix representation) of the metaplectic group (40).

The action of the Majorana Weyl matrices defined above will lead to

$$\sigma_{\alpha}\psi(t) = F^{1/2}(t)\left(v_{ab}^{1/2}\right)^{*}, \tag{43}$$

$$\sigma_{\beta}\psi(t) = -F^{1/2}(t)\left(w_{ab}^{1/2}\right)^{*}, \tag{44}$$

$$\sigma_{\gamma}\psi(t) = F^{1/2}(t)\left(w_{ab}^{1/2}\right) \tag{45}$$

The projection relations can be put in a more compact form

$$\psi^{*}\sigma_{\alpha}\psi = F(t)\,2Re(\alpha), \tag{46}$$

$$\psi^{*}\sigma_{\beta}\psi = F(t)\,2Im(\alpha), \tag{47}$$

$$\psi^{*}\sigma_{\gamma}\psi = 0. \tag{48}$$

$$|\psi|^{2} = F(t)\,2|\alpha|. \tag{49}$$

A. Group manifold → space-time reduction

Using the ingredients above, the projections are reduced to simple expressions that bring us the specific splitting of the eigenvalue of the a operator (q and p)

$$\psi^* \sigma_\alpha \psi = |\psi|^2 \frac{Re(\alpha)}{|\alpha|} \rightarrow \text{position} \tag{50}$$

$$\psi^* \sigma_\beta \psi = |\psi|^2 \frac{Im(\alpha)}{|\alpha|} \rightarrow \text{momentum} \tag{51}$$

$$\psi^* \sigma_\gamma \psi = 0. \tag{52}$$

By means of the basic coherent states, the generator of the 6-dimensional $Mp(4)$ breaks down to the 4-dimensional (observable) spacetime.

$$\boxed{Mp(4)} \quad \xrightarrow[\text{non-observable}]{\psi} \quad \boxed{\begin{array}{c} g_{ab} \\ \text{observable(physical)} \end{array}}$$

And this is not trivial: as explained in several references [1,9,10 and references therein] the physical states are the projections of the generators of the 6-dimensional $Mp(4)$ on the 4-dimensional spacetime by means of the (unnobservable) coherent states of fractional spin. And the normalized spinors have a total projection in terms of normalized coherent states

$$\frac{\psi^* \vec{\sigma} \psi}{|\psi|^2} = \frac{\alpha}{|\alpha|}. \tag{53}$$

However, in the general case the polar decomposition can be introduced as $\alpha = |\alpha| e^{i\theta}$, picking up the explicit phase of the coherent state (and having precisely the square of the norm of the state such an order parameter)

$$\frac{\psi^* \vec{\sigma} \psi}{|\psi|^2} = e^{i\theta}. \tag{54}$$

B. Symmetry transformations

We now use the faithful real representation of the matrices previously introduced as generators of the symmetry transformations. From the Majorana representation of the operators and the previous results we have in matrix form

$$g_{ab} \rightarrow g_{ab}\big|_i = e^{-i\sigma^i \varphi_i(t)} g_{ab} \tag{55}$$

$$\psi \rightarrow \psi^\mu\big|_i = e^{-i\sigma^i \varphi_i(t)} \psi \tag{56}$$

where $\varphi_i(t)$ is the generic parameter associated to the action of the generators (40) given the specific rotation (or boost). They can be labeled $\alpha\,(t)$, $\beta\,(t)$ and $\gamma\,(t)$.with the corresponding symmetry of the associated generator.

We can explicitly write expressions above in the following way

$$\begin{aligned} g_{ab}\big|_{\alpha} &= \cos\left(\alpha(t)\right) g_{ab} - i \sin\left(\alpha(t)\right) \sigma^{\mu} g_{ab} && (57) \\ \psi\big|_{\alpha} &= \cos\left(\alpha(t)\right) \psi - i \sin\left(\alpha(t)\right) \sigma^{\mu} \psi. && (58) \end{aligned}$$

and similarly for γ. On the other hand, as σ^{β} has an anti hermitian representation, it acts as a boost (hyperbolic rotation)

$$\begin{aligned} g_{ab}\big|_{\beta} &= \cosh\left(\beta(t)\right) g_{ab} - \sinh\left(\beta(t)\right) \sigma^{\mu} g_{ab} && (59) \\ \psi\big|_{\beta} &= \cosh\left(\beta(t)\right) \psi - \sinh\left(\beta(t)\right) \sigma^{\mu} \psi && (60) \end{aligned}$$

Then, we can consider the observable state affected with an arbitrary phase

$$g_{ab}(t) = e^{-i\varphi(t)} F\,(t)\, v_{ab} \tag{61}$$

the square (observable) state under the action of σ^{α} oscillates harmonically with respect of the "natural complex spinors" v_{ab} and w_{ab} as

$$g_{ab}^{(1)}(t) = F\,(t) \left[\cos\left(\alpha(t)\right) v_{ab} - i \sin\left(\alpha(t)\right) v_{ab}^{*}\right] \tag{62}$$

the square (observable) under the action of σ^{β} is "boosted" (hyperbolic rotation) as

$$g_{ab}^{(2)}(t) = F\,(t) \left[\cosh\left(\beta(t)\right) v_{ab} - \sinh\left(\beta(t)\right) w_{ab}^{*}\right]$$

Finally under the action of σ^{γ} we have

$$g_{ab}^{(3)}(t) = F\,(t) \left[\cos\left(\gamma(t)\right) v_{ab} - i \sin\left(\gamma(t)\right) w_{ab}\right]$$

Similar action of the transformation operators can be easily realized over the square root states: we only need to change $v_{ab} \to v_{ab}^{1/2}$, etc. Here is quite evident the important fundamental role of the"natural complex spinors" v_{ab} and w_{ab} : all the action of the group generators of the underlying maniflod over the states are clearly described only in this preferred spinor basis,(i.e. eqs.(41,42)).

V. CONCLUDING REMARKS

In this paper we have analyzed several general aspects of the model introduced by Volkov and Pashnev in [3], and correctly interpreted from the quantum and field theoretical points of view in [1,2,10]. The model is characterized by a $N = 1$ superspace equipped with a non-degenerate supermetric.

We explicitly shown that, in full agreement with the Casalbuoni's $G4$ model [4], the Lagrangian must be of the form of a measure: *i.e.*, the square root of the super-line element, in order to preserv, physically and mathematically speaking, the classical relativistic symmetries (as in the case of the classical relativistic particle), and also the fact that the supersymmetry appears as a pure relativistic effect[4].

From the geometrical point of view, the super-line element (non-degenerate supermetric) induces the square of the supervelocity that must be constrained to $-c^2$ (as in the classical relativistic case) considering the square of the supermomentum that fulfil automatically the mass shell condition. This fact leads to a deviation of the 4-impulse (mass constraint) that can be mechanically interpreted as a modification of the Newton's law. The possibility of antigravitational effects, alternatives to dark matter and the relation with the MOND conjecture and the conservation of the CPT symmetry in such a case is under study [7].

With respect to the solutions obtained, we have confirmed from another point of view the results of refs.[1,2,10] that there are two types of states: the basic (non-observable) ones and observable physical states. The basic states are coherent states corresponding to the double covering of the $SL(2C)$ or the metaplectic group [1,2,9,10] responsible for projecting the symmetries of the 6 dimensional $Mp(4)$ group space to the 4 dimensional spacetime by means of a bilinear combination of the $Mp(4)$ generators.

An important new result we have found is that there exist an oscillatory fermionic effect in the B_0 part of the supermultiplet as a Zitterwebegung, but between the chiral and antichiral components of this Majorana bispinor. This effect has been, as far as we know, never mentioned in the literature. This issue will be treated in detail in a future work [7,8].

The temporal and special limits were explicitly expressed through the observable and non-observable states and their corresponding probability currents. Again, as pointed out in ref.[2], the $|a|$ plays the main role in the restoration of the chiral symmetry, as is easily seen from the boson ultralocalization limit and the fermionic bosonization limits. . Quantum

aspects of the physical states, the basic ones and the projection relation between them, were completely described due the introduction of a new Majorana-Weyl representation of the generators of the underlying group manifold. Also we find that the action of these generators over the states vaccum solution of the model, are clearly described only in a preferred spinor basis given by eqs.(41,42).

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